• Classical Sets and Fuzzy Sets
• Classical Sets
• Operation on Classical Sets
• Properties of Classical (Crisp) Sets
• Mapping of Classical Sets to Functions

• Fuzzy Sets
• Notation Convention for Fuzzy Sets
• Fuzzy Set Operations
A classical set is defined by **crisp boundaries**

A fuzzy set is prescribed by **vague** or **ambiguous** properties; hence its boundaries are ambiguously specified

**X** (Universe of discourse)
Classical Sets

The universe of discourse is the universe of all available information on a given problem. A universe of discourse, $X$, as a collection of objects all having the same characteristics:

- The clock speeds of computer CPUs
- The operating currents of an electronic motor
- The operating temperature of a heat pump (in degrees Celsius)
- The Richter magnitudes of an earthquake
- The integers 1 to 10

The individual elements in the universe $X$ will be denoted as $x$. The features of the elements in $X$ can be discrete, countable integers or continuous valued quantities on the real line.

The total number of elements in a universe $X$ is called its cardinal number, denoted $n_x$. 
Classical Sets

Collections of elements within a universe are called **sets**

universe of discourse: The Richter magnitudes of an earthquake
*Set in the universe of discourse?*

Collections of elements within sets are called **subsets**

The collection of all possible sets in the universe is called the **whole set (power set)**.
Classical Sets

We have a universe comprised of three elements, \( X = \{a, b, c\} \)

The cardinal number, \( n_x \)?

*The power set, \( P(X) \)?*

The cardinality of the power set?
Operation on Classical Sets

**Union**

\[ A \cup B = \{x \mid x \in A \text{ or } x \in B\} \]

The union between the two sets, denoted \( A \cup B \), represents all those elements in the universe that reside in (or belong to) the set \( A \), the set \( B \), or both sets \( A \) and \( B \).

This operation is also called the **logical or**

Union of sets \( A \) and \( B \) (logical or) in terms of Venn diagrams
Operation on Classical Sets

**Intersection** \[ A \cap B = \{x \mid x \in A \text{ and } x \in B\} \]

The intersection of the two sets, denoted \( A \cap B \), represents all those elements in the universe \( X \) that **simultaneously** reside in (or belong to) both sets \( A \) and \( B \).

This operation is also called the **logical and**

Intersection of sets A and B.
The **complement** of a set $A$, is defined as the collection of all elements in the universe that do not reside in the set $A$. 

$$\overline{A} = \{ x \mid x \not\in A, x \in X \}$$
Operation on Classical Sets

$\text{Difference}$

$A \setminus B = \{ x \mid x \in A \text{ and } x \notin B \}$

The **difference** of a set $A$ with respect to $B$, denoted $A \setminus B$, is defined as the collection of all elements in the universe that reside in $A$ and that do not reside in $B$ simultaneously.

![Difference operation A \setminus B](image)
Properties of Classical (Crisp) Sets

Commutativity  
\[ A \cup B = B \cup A \]
\[ A \cap B = B \cap A \]

Associativity  
\[ A \cup (B \cup C) = (A \cup B) \cup C \]
\[ A \cap (B \cap C) = (A \cap B) \cap C \]

Distributivity  
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
\[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \] (2.7)

Idempotency  
\[ A \cup A = A \]
\[ A \cap A = A \]

Identity  
\[ A \cup \emptyset = A \]
\[ A \cap X = A \]
\[ A \cap \emptyset = \emptyset \]
\[ A \cup X = X \]

Transitivity  
If \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \)
Properties of Classical (Crisp) Sets

Two special properties of set operations,

The excluded middle axioms
De Morgan’s principles
Properties of Classical (Crisp) Sets

The excluded middle axioms

not valid for both classical sets and fuzzy sets.

There are two excluded middle axioms
The first, called the axiom of the excluded middle, deals with the union of a set $A$ and its complement,
the second, called the axiom of contradiction, represents the intersection of a set $A$ and its complement.

\[
\begin{align*}
\text{Axiom of the excluded middle} & : A \cup \overline{A} = X \\
\text{Axiom of the contradiction} & : A \cap \overline{A} = \emptyset
\end{align*}
\]
Properties of Classical (Crisp) Sets

*De Morgan’s principles*

\[
\overline{A \cap B} = \overline{A} \cup \overline{B} \\
\overline{A \cup B} = \overline{A} \cap \overline{B}
\]

information about the complement of a set (or event), or the complement of combinations of sets (or events), rather than information about the sets themselves
Properties of Classical (Crisp) Sets

*De Morgan’s principles*

**Example:** A shallow arch consists of two slender members as shown in Fig. If either member fails, then the arch will collapse.

\[ E_1 = \text{survival of member 1} \]
\[ E_2 = \text{survival of member 2} \]

Survival of the arch = ?
Collapse of the arch = ?

Logically, collapse of the arch will occur if either of the members fails
Properties of Classical (Crisp) Sets

**De Morgan’s principles**

**Example 1:** A shallow arch consists of two slender members as shown in Fig. If either member fails, then the arch will collapse.

E1 = survival of member 1 and
E2 = survival of member 2,

Survival of the arch = \( E_1 \cap E_2 \)

Collapse of the arch = \( \overline{E_1 \cap E_2} \).

Collapse of the arch will occur if either of the members fails:

\( \overline{E_1 \cup \overline{E_2}} \).

Illustration of De Morgan’s principle:

\( \overline{E_1 \cap E_2} = \overline{E_1} \cup \overline{E_2} \)
Mapping is an important concept in relating set-theoretic forms to function-theoretic representations of information.

In its most general form it can be used to map elements or subsets on one universe of discourse to elements or sets in another universe.

If an element \( x \) is contained in \( X \) and corresponds to an element \( y \) contained in \( Y \), it is generally termed a mapping from \( X \) to \( Y \),

\[ f : X \rightarrow Y \]
The characteristic (indicator) function $\chi_A$ is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

Membership function is a mapping for crisp set A.
Mapping of Classical Sets to Functions

Example: a universe with three elements, \( X = \{a, b, c\} \), we desire to map the elements of the power set of \( X \), i.e., \( P(X) \), to a universe, \( Y \), consisting of only two elements (the characteristic function), \( Y = \{0, 1\} \)

the elements of the power set?

the elements in the value set \( V(P(X)) \)?
Example: a universe with three elements, $X = \{a, b, c\}$, we desire to map the elements of the power set of $X$, i.e., $P(X)$, to a universe, $Y$, consisting of only two elements (the characteristic function), $Y = \{0, 1\}$

- the elements of the power set
  $P(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

- the elements in the value set $V(P(X))$
  $V(P(X)) = \{\{0, 0, 0\}, \{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}, \{1, 1, 0\}, \{0, 1, 1\}, \{1, 0, 1\}, \{1, 1, 1\}\}$
Mapping of Classical Sets to Functions

The **union** of these two sets in terms of **function-theoretic terms** is given as follows (the symbol $\lor$ is the maximum):

$$\text{Union } A \cup B \rightarrow \chi_{A \cup B}(x) = \chi_A(x) \lor \chi_B(x) = \max(\chi_A(x), \chi_B(x))$$

The **intersection** of these two sets in **function-theoretic** terms is given by (the symbol $\land$ is the minimum operator):

$$\text{Intersection } A \cap B \rightarrow \chi_{A \cap B}(x) = \chi_A(x) \land \chi_B(x) = \min(\chi_A(x), \chi_B(x))$$

The **complement** of a single set on universe $X$, say $A$, is given by

$$\text{Complement} \quad \overline{A} \rightarrow \chi_{\overline{A}}(x) = 1 - \chi_A(x)$$

Example
The boundaries of the fuzzy sets are vague and ambiguous. Hence, membership of an element from the universe in this set is measured by a function that attempts to describe vagueness and ambiguity.
Fuzzy Sets

the set of tall people
Fuzzy Sets

degree of membership, $\mu$

sharp-edged membership function for TALL

tall ($\mu = 1.0$)

not tall ($\mu = 0.0$)

definitely a tall person ($\mu = 0.95$)

really not very tall at all ($\mu = 0.30$)

height

continuous membership function for TALL

You must be taller than this line to be considered TALL
Fuzzy Sets

Elements of a fuzzy set are mapped to a universe of membership values using a function-theoretic form.

Fuzzy sets are denoted by a set symbol with a tilde understrike; \( A_\sim \) would be the fuzzy set \( A \).

This function maps elements of a fuzzy set \( A_\sim \) to a real numbered value on the interval 0 to 1.
If an element in the universe, say \( x \), is a member of fuzzy set \( A_\sim \), then this mapping is given by

\[
\mu_{A_\sim}(x) \in [0,1].
\]
Notation Convention for Fuzzy Sets

When the universe of discourse, X, is discrete and finite, is as follows for a fuzzy set $A_{\sim}$:

$$A = \left\{ \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots \right\} = \left\{ \sum_i \frac{\mu_A(x_i)}{x_i} \right\}$$

When the universe, X, is continuous and infinite, the fuzzy set $A_{\sim}$

$$A = \left\{ \int \frac{\mu_A(x)}{x} \right\}$$

Membership function for fuzzy set $A_{\sim}$

Example
Fuzzy Set Operations

Three fuzzy sets $A$, $B$, and $C$ on the universe $X$

For a given element $x$ of the universe, the following function-theoretic operations for the set-theoretic operations of union, intersection, and complement are defined for $A$, $B$, and $C$ on $X$

### Standard fuzzy operations

- **Union**
  \[
  \mu_{A \cup B}(x) = \mu_A(x) \lor \mu_B(x)
  \]

- **Intersection**
  \[
  \mu_{A \cap B}(x) = \mu_A(x) \land \mu_B(x)
  \]

- **Complement**
  \[
  \mu_{\overline{A}}(x) = 1 - \mu_A(x)
  \]
Fuzzy Set Operations

Union of fuzzy sets $A_\sim$ and $B_\sim$

Intersection of fuzzy sets $A_\sim$ and $B_\sim$

Complement of fuzzy sets $A_\sim$ and $B_\sim$
Fuzzy Set Operations

All other operations on classical sets also hold for fuzzy sets, except for the excluded middle axioms

\[ A \cup \overline{A} \neq X \]
\[ A \cap \overline{A} \neq \emptyset \]

Proof, classical and fuzzy sets?
Examples of Fuzzy Set Operations

Example: chemical engineering case
Suppose the selection of an appropriate analyzer to monitor the “sales gas” sour gas concentration is important. This selection process can be complicated by the fact that one type of analyzer, say A, does not provide an average suitable pressure range but it does give a borderline value of instrument dead time; in contrast another analyzer, say B, may give a good value of process dead time but a poor pressure range.
Suppose for this problem we consider three analyzers: A, B and C.

1. the pressure range suitability of analyzers A, B, and C (a membership of 0 is not suitable, a value of 1 is excellent) ?

2. the instrument dead time suitability of analyzers A, B, and C (again, 0 is not suitable and 1 is excellent) ?

3. the analyzers that are not suitable for pressure range and instrument dead time, respectively ?

4. which analyzer is most suitable in either category ?

5. which analyzer is suitable in both categories ?
Examples of Fuzzy Set Operations

Example:
We are asked to select an implementation technology for a numerical processor. Computation throughput is directly related to clock speed. We are considering whether the design should be implemented using medium-scale integration (MSI) with discrete parts, field-programmable array parts (FPGA), or multichip modules (MCM).

Define the universe of potential clock speeds as MHz; and define MSI, FPGA, and MCM as fuzzy sets of clock frequencies that should be implemented in each of these technologies. The following table defines the membership values for each of the three fuzzy sets.

<table>
<thead>
<tr>
<th>Clock frequency, MHz</th>
<th>MSI</th>
<th>FPGA</th>
<th>MCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.3</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.7</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>0.4</td>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>0.5</td>
<td>0.7</td>
</tr>
<tr>
<td>80</td>
<td>0</td>
<td>0.2</td>
<td>1</td>
</tr>
<tr>
<td>100</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Representing the three sets as MSI = M, FPGA = F, and MCM = C, find the following:
Examples of Fuzzy Set Operations

Example:
Samples of a new microprocessor IC chip are to be sent to several customers for beta testing. The chips are sorted to meet certain maximum electrical characteristics, say frequency and temperature rating, so that the “best” chips are distributed to preferred customer 1. Suppose that each sample chip is screened and all chips are found to have a maximum operating frequency in the range 7–15 MHz at 20°C. Also, the maximum operating temperature range (20°C ± T ) at 8 MHz is determined. Suppose there are eight sample chips with the following electrical characteristics:

<table>
<thead>
<tr>
<th>Chip number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{\text{max}}$, MHz</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>$\Delta T_{\text{max}}$, °C</td>
<td>0</td>
<td>0</td>
<td>20</td>
<td>40</td>
<td>30</td>
<td>50</td>
<td>40</td>
<td>60</td>
</tr>
</tbody>
</table>

The following fuzzy sets are defined:

- $A = \text{set of “fast” chips} = \text{chips with } f_{\text{max}} \geq 12 \text{ MHz}$
- $\tilde{A} = \text{set of “slow” chips} = \text{chips with } f_{\text{max}} \geq 8 \text{ MHz}$
- $\overline{C} = \text{set of “cold” chips} = \text{chips with } \Delta T_{\text{max}} \geq 10 \degree \text{C}$
- $D = \text{set of “hot” chips} = \text{chips with } \Delta T_{\text{max}} \geq 50 \degree \text{C}$