CLASSICAL LOGIC and FUZZY LOGIC
In classical logic, a simple proposition $P$ is a linguistic, or declarative, statement contained within a universe of elements, $X$, that can be identified as being a collection of elements in $X$ that are strictly true or strictly false.

The veracity (truth) of an element in the proposition $P$ can be assigned a binary truth value, called $T(P)$,

For binary (Boolean) classical logic, $T(P)$ is assigned a value of 1 (truth) or 0 (false).

If $U$ is the universe of all propositions, then $T$ is a mapping of the elements, $u$, in these propositions (sets) to the binary quantities $(0, 1)$, or

$$T : u \in U \rightarrow (0, 1)$$
Example 5.1. Let P be the proposition “The structural beam is an 18WF45” and let Q be the proposition “The structural beam is made of steel.” Let X be the universe of structural members comprised of girders, beams, and columns; x is an element (beam), A is the set of all wide-flange (WF) beams, and B is the set of all steel beams. Hence,

\[ P : x \text{ is in } A \]
\[ Q : x \text{ is in } B \]
Let P and Q be two simple propositions on the same universe of discourse that can be combined using the following five logical connectives:

- Disjunction (V)
- Conjunction (∧)
- Negation (¬)
- Implication (→)
- Equivalence (↔)
define sets $A$ and $B$ from universe, where these sets might represent linguistic ideas or thoughts.

A propositional calculus (sometimes called the algebra of propositions) will exist for the case where proposition $P$ measures the truth of the statement that an element, $x$, from the universe $X$ is contained in set $A$ and the truth of the statement $Q$ that this element, $x$, is contained in set $B$, or more conventionally,

$$P : \text{truth that } x \in A$$
$$Q : \text{truth that } x \in B$$

where truth is measured in terms of the truth value, i.e.,

- If $x \in A$, $T(P) = 1$; otherwise, $T(P) = 0$
- If $x \in B$, $T(Q) = 1$; otherwise, $T(Q) = 0$

or, using the characteristic function to represent truth (1) and falsity (0), the following notation results:

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \not\in A \end{cases}$$
The five logical connectives already defined can be used to create **compound propositions**, where a compound proposition is defined as a logical proposition formed by logically connecting two or more simple propositions.

**Disjunction**

\[ P \lor Q : x \in A \text{ or } x \in B \]

Hence, \( T(P \lor Q) = \max(T(P), T(Q)) \)

**Conjunction**

\[ P \land Q : x \in A \text{ and } x \in B \]

Hence, \( T(P \land Q) = \min(T(P), T(Q)) \)

**Negation**

If \( T(P) = 1 \), then \( T(\overline{P}) = 0 \); if \( T(P) = 0 \), then \( T(\overline{P}) = 1 \).

**Implication**

\[ (P \rightarrow Q) : x \not\in A \text{ or } x \in B \]

Hence, \( T(P \rightarrow Q) = T(\overline{P} \cup Q) \)

**Equivalence**

\[ (P \leftrightarrow Q) : T(P \leftrightarrow Q) = \begin{cases} 1, & \text{for } T(P) = T(Q) \\ 0, & \text{for } T(P) \neq T(Q) \end{cases} \]
Truth table for various compound propositions

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>$\overline{P}$</th>
<th>$P \lor Q$</th>
<th>$P \land Q$</th>
<th>$P \rightarrow Q$</th>
<th>$P \leftrightarrow Q$</th>
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The implication $P \rightarrow Q$ can be represented in set-theoretic terms by the relation $R$,

Suppose the implication operation involves two different universes of discourse; $P$ is a proposition described by set $A$, which is defined on universe $X$, and $Q$ is a proposition described by set $B$, which is defined on universe $Y$.

\[
R = (A \times B) \cup (\overline{A} \times Y) \equiv \text{IF } A, \text{ THEN } B
\]

\[
\text{IF } x \in A \text{ where } x \in X \text{ and } A \subseteq X
\]

\[
\text{THEN } y \in B \text{ where } y \in Y \text{ and } B \subseteq Y
\]
This **implication** is also equivalent to the **linguistic rule form**, 
**IF A, THEN B.**

\[
P \rightarrow Q : \text{IF } x \in A, \text{ THEN } y \in B, \quad \text{or} \quad P \rightarrow Q \equiv \bar{A} \cup B
\]

Another compound proposition in linguistic rule form is the expression

**IF A, THEN B, ELSE C**

The set-theoretic equivalent of this compound proposition is given by

\[
\text{IF A, THEN B, ELSE C } \equiv (A \times B) \cup (\bar{A} \times C) = R = \text{ relation on } X \times Y
\]
Tautologies

In classical logic it is useful to consider compound propositions that are always true, irrespective of the truth values of the individual simple propositions.

Classical logical compound propositions with this property are called tautologies.

Tautologies are useful for deductive reasoning, for proving theorems, and for making deductive inferences.
CLASSICAL LOGIC

Tautologies

Some common tautologies follow:

\[ B \cup B \leftrightarrow X \]
\[ A \cup X; \quad \overline{A} \cup X \leftrightarrow X \]
\[ (A \land (A \rightarrow B)) \rightarrow B \quad (\text{modus ponens}) \]
\[ (\overline{B} \land (A \rightarrow B)) \rightarrow \overline{A} \quad (\text{modus tollens}) \]

Proof
### Classical Logic

**Tautologies**

**Truth table (modus ponens)**

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Tautology

**Truth table (modus tollens)**

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Tautology
Deductive Inferences
The *modus ponens deduction* is used as a tool for making inferences in rule-based systems. A typical if–then rule is used to determine whether an antecedent (cause or action) infers a consequent (effect or reaction).

Suppose we have a rule of the form IF A, THEN B, where A is a set defined on universe X and B is a set defined on universe Y. As discussed before, this rule can be translated into a relation between sets A and B;

\[ R = (A \times B) \cup (\overline{A} \times Y) \]
Deductive Inferences

Suppose a new antecedent, say $A'$, is known. Can we use modus ponens deduction to infer a new consequent, say $B'$, resulting from the new antecedent? That is, can we deduce, in rule form, IF $A'$, THEN $B'$?

Yes, through the use of the composition operation. Since “$A$ implies $B$” is defined on the Cartesian space $X \times Y$, $B$ can be found through the following set-theoretic formulation,

$$B' = A' \circ R = A' \circ ((A \times B) \cup (\overline{A} \times Y))$$
Deductive Inferences

The rule IF A, THEN B (proposition P is defined on set A in universe X, and proposition Q is defined on set B in universe Y), i.e., \((P \rightarrow Q) = R = (A \times B) \cup (\overline{A} \times Y)\), is then defined in function-theoretic terms as

\[
\chi_R(x, y) = \max[(\chi_A(x) \land \chi_B(y)), ((1 - \chi_A(x)) \land 1)]
\]

where \(\chi(\ )\) is the characteristic function as defined before.
Deductive Inferences

The compound rule IF A, THEN B, ELSE C can also be defined in terms of a matrix relation as

\[ R = (A \times B) \cup (\overline{A} \times C) \Rightarrow (P \rightarrow Q) \land (\overline{P} \rightarrow S) \]

where the membership function is determined as

\[ \chi_R(x, y) = \max[(\chi_A(x) \land \chi_B(y)), ((1 - \chi_A(x)) \land \chi_C(y))] \]
CLASSICAL LOGIC

EXAMPLE

Suppose we have two universes of discourse for a heat exchanger problem described by the following collection of elements,
X = {1, 2, 3, 4} and
Y = {1, 2, 3, 4, 5, 6}.
Suppose X is a universe of normalized temperatures and Y is a universe of normalized pressures.

Define crisp set A on universe X and crisp set B on universe Y as follows:
A = {2, 3} and
B = {3, 4}.

The deductive inference IF A, THEN B (i.e., IF temperature is A, THEN pressure is B) will yield a matrix describing the membership values of the relation R, i.e., \( \chi_R(x, y) \)
That is, the matrix R represents the rule IF A, THEN B as a matrix of characteristic (crisp membership) values.

\[
B' = A' \circ R = A' \circ ((A \times B) \cup (\overline{A} \times Y))
\]
The restriction of classical propositional calculus to a two-valued logic has created many interesting paradoxes over the ages.

For example, the Barber of Seville is a classic paradox (also termed Russell’s barber). *In the small Spanish town of Seville, there is a rule that all and only those men who do not shave themselves are shaved by the barber. Who shaves the barber?*

Another example comes from ancient Greece. *Does the liar from Crete lie when he claims, “All Cretians are liars?” If he is telling the truth, his statement is false. But if his statement is false, he is not telling the truth. A simpler form of this paradox is the two-word proposition, “I lie.”*

The statement can not be both true and false.
A fuzzy logic proposition, $P_\sim$, is a statement involving some concept without clearly defined boundaries.

Most natural language is fuzzy, in that it involves vague and imprecise terms. Statements describing a person’s height or weight or assessments of people’s preferences about colors or menus can be used as examples of fuzzy propositions.

The truth value assigned to $P_\sim$ can be any value on the interval $[0, 1]$. The assignment of the truth value to a proposition is actually a mapping from the interval $[0, 1]$ to the universe $U$ of truth values, $T$, as indicated

$$T : u \in U \rightarrow (0, 1)$$
FUZZY LOGIC

As in classical binary logic, we assign a logical proposition to a set in the universe of discourse. Fuzzy propositions are assigned to fuzzy sets. Suppose proposition $\mathcal{P}$ is assigned to fuzzy set $A$; then the truth value of a proposition, denoted $T(\mathcal{P})$, is given by

$$T(\mathcal{P}) = \mu_A(x) \quad \text{where} \quad 0 \leq \mu_A \leq 1$$

indicates that the degree of truth for the proposition $\mathcal{P}: x \in A$ is equal to the membership grade of $x$ in the fuzzy set $A$. 

FUZZY LOGIC

The **logical connectives** of negation, disjunction, conjunction, and implication are also defined for a fuzzy logic.

**Negation**

\[ T(\overline{P}) = 1 - T(P) \]

**Disjunction**

\[ P \lor Q : x \text{ is } A \text{ or } B \quad T(P \lor Q) = \max(T(P), T(Q)) \]

**Conjunction**

\[ P \land Q : x \text{ is } A \text{ and } B \quad T(P \land Q) = \min(T(P), T(Q)) \]

**Implication [Zadeh, 1973]**

\[ \overline{P} \rightarrow Q : x \text{ is } A, \text{ then } x \text{ is } B \]

\[ T(\overline{P} \rightarrow Q) = T(\overline{P} \lor Q) = \max(T(\overline{P}), T(Q)) \]
FUZZY LOGIC

As before in binary logic, the implication connective can be modeled in rule-based form;

\[ P \rightarrow Q \text{ is, IF } x \text{ is } \tilde{A}, \text{ THEN } y \text{ is } \tilde{B} \]

and it is equivalent to the following fuzzy relation,

\[ R = (\tilde{A} \times \tilde{B}) \cup (\tilde{A} \times Y) \]

The membership function of \( R_{\sim} \) is expressed by the following formula:

\[ \mu_R(x, y) = \max[(\mu_{\tilde{A}}(x) \land \mu_{\tilde{B}}(y)), (1 - \mu_{\tilde{A}}(x))] \]
FUZZY LOGIC

When the logical conditional implication is of the compound form

\[ \text{IF } x \text{ is } A, \text{ THEN } y \text{ is } B, \text{ ELSE } y \text{ is } C \]

then the equivalent fuzzy relation, \( R \), is expressed as \( R = (A \times B) \cup (\overline{A} \times C) \), in a form whose membership function is expressed by the following formula:

\[
\mu_R(x, y) = \max \left[ (\mu_A(x) \land \mu_B(y)), ((1 - \mu_A(x)) \land \mu_C(y)) \right]
\]
Example 5.9. Suppose we are evaluating a new invention to determine its commercial potential. We will use two metrics to make our decisions regarding the innovation of the idea. Our metrics are the “uniqueness” of the invention, denoted by a universe of novelty scales, \( X = \{1, 2, 3, 4\} \), and the “market size” of the invention’s commercial market, denoted on a universe of scaled market sizes, \( Y = \{1, 2, 3, 4, 5, 6\} \). In both universes the lowest numbers are the “highest uniqueness” and the “largest market,” respectively. A new invention in your group, say a compressible liquid of very useful temperature and viscosity conditions, has just received scores of “medium uniqueness,” denoted by fuzzy set \( \tilde{A} \), and “medium market size,” denoted fuzzy set \( \tilde{B} \). We wish to determine the implication of such a result, i.e., IF \( \tilde{A} \), THEN \( \tilde{B} \). We assign the invention the following fuzzy sets to represent its ratings:

\[
\begin{align*}
\tilde{A} &= \text{medium uniqueness} = \left\{ \frac{0.6}{2} + \frac{1}{3} + \frac{0.2}{4} \right\} \\
\tilde{B} &= \text{medium market size} = \left\{ \frac{0.4}{2} + \frac{1}{3} + \frac{0.8}{4} + \frac{0.3}{5} \right\} \\
\zeta &= \text{diffuse market size} = \left\{ \frac{0.3}{1} + \frac{0.5}{2} + \frac{0.6}{3} + \frac{0.6}{4} + \frac{0.5}{5} + \frac{0.3}{6} \right\}
\end{align*}
\]

\[
\tilde{R} = (\tilde{A} \times \tilde{B}) \cup (\overline{\tilde{A}} \times Y)
\]
FUZZY LOGIC

Approximate reasoning
The ultimate goal of fuzzy logic is to form the theoretical foundation for reasoning about imprecise propositions; such reasoning has been referred to as approximate reasoning [Zadeh, 1976, 1979].

Approximate reasoning is analogous to classical logic for reasoning with precise propositions, and hence is an extension of classical propositional calculus that deals with partial truths.
FUZZY LOGIC

Approximate reasoning

Suppose we have a rule-based format to represent fuzzy information. These rules are expressed in conventional antecedent-consequent form, such as

Rule 1: IF $x$ is $A$, THEN $y$ is $B$, where $A$ and $B$ represent fuzzy propositions (sets).

Now suppose we introduce a new antecedent, say $A'$, and we consider the following rule:

Rule 2: IF $x$ is $A'$, THEN $y$ is $B'$.

From information derived from Rule 1, is it possible to derive the consequent in Rule 2, $B'$? The answer is yes, and the procedure is fuzzy composition. The consequent $B'$ can be found from the composition operation, $B' = A' \circ R$. 
Example 5.11. For research on the human visual system, it is sometimes necessary to characterize the strength of response to a visual stimulus based on a magnetic field measurement or on an electrical potential measurement. When using magnetic field measurements, a typical experiment will require nearly 100 off/on presentations of the stimulus at one location to obtain useful data. If the researcher is attempting to map the visual cortex of the brain, several stimulus locations must be used in the experiments. When working with a new subject, a researcher will make preliminary measurements to determine if the type of stimulus being used evokes a good response in the subject. The magnetic measurements are in units of femtotesla ($10^{-15}$ tesla). Therefore, the inputs and outputs are both measured in terms of magnetic units.

We will define inputs on the universe $X = [0, 50, 100, 150, 200]$ femtotesla, and outputs on the universe $Y = [0, 50, 100, 150, 200]$ femtotesla. We will define two fuzzy sets, two different stimuli, on universe $X$:

\[
\tilde{W} = \text{“weak stimulus”} = \left\{ \frac{1}{0} + \frac{0.9}{50} + \frac{0.3}{100} + \frac{0}{150} + \frac{0}{200} \right\} \subset X
\]

\[
\tilde{M} = \text{“medium stimulus”} = \left\{ \frac{0}{0} + \frac{0.4}{50} + \frac{1}{100} + \frac{0.4}{150} + \frac{0}{200} \right\} \subset X
\]

and one fuzzy set on the output universe $Y$,

\[
\tilde{S} = \text{“severe response”} = \left\{ \frac{0}{0} + \frac{0}{50} + \frac{0.5}{100} + \frac{0.9}{150} + \frac{1}{200} \right\} \subset Y
\]

We will construct the proposition: IF “weak stimulus” THEN not “severe response,” using classical implication.

\[
\text{IF } \tilde{W} \text{ THEN } \tilde{S} = \tilde{W} \rightarrow \tilde{S} = (\tilde{W} \times \tilde{S}) \cup (\tilde{W} \times Y)
\]